

# Theory Including Future Not Excluded

## — Formulation of Complex Action Theory II —

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### Abstract

We study a complex action theory (CAT) whose path runs over not only past but also future. We show that if we regard a matrix element defined in terms of the future state at time  $T_B$  and the past state at time  $T_A$  as an expectation value in the CAT, then we are allowed to have the Heisenberg equation, the Ehrenfest's theorem and the conserved probability current density. In addition we show that the expectation value at the present time  $t$  of a future-included theory for large  $T_B - t$  and large  $t - T_A$  corresponds to that of a future-not-included theory with a proper inner product for large  $t - T_A$ . Hence the CAT with future explicitly present in the formalism and influencing in principle the past is not excluded phenomenologically, because the effects are argued to be very small in the present era. Furthermore we explicitly derive the Hamiltonian for the future state via path integral, and confirm that it is given by the hermitian conjugate of the Hamiltonian for the past state.

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## §1. Introduction

Quantum theories are properly formulated via the Feynman path integral (FPI). Usually the action  $S$  is real, and it is thought to be more fundamental than the integrand  $\exp(\frac{i}{\hbar}S)$ . However, if we assume that the integrand is more fundamental than the action in quantum theory, then it is naturally thought that since the integrand is complex, the action could also be complex. Based on this speculation and other related works in some backward causation developments inspired by general relativity<sup>1)</sup> and the non-locality explanation of fine-tuning problems,<sup>2)</sup> the complex action theory (CAT) has so far been studied intensively.<sup>3),4)</sup> The imaginary part of the action is thought to give some falsifiable predictions, and many interesting suggestions have been made for the Higgs mass,<sup>5)</sup> quantum-mechanical philosophy,<sup>6)-8)</sup> some fine-tuning problems,<sup>9),10)</sup> black holes,<sup>11)</sup> the de Broglie-Bohm particle, and a cut-off in loop diagrams.<sup>12)</sup>

In refs.<sup>3)-12)</sup> they studied a future-included version, i.e., the theory including not only a past time but also a future time as an integration interval of time. In contrast to these works, in refs.<sup>13)-15)</sup> we studied a future-not-included version. In ref.<sup>13)</sup> we analyzed the time development of some state by a non-Hermitian diagonalizable bounded Hamiltonian  $H$ , and showed that we can effectively obtain a Hermitian Hamiltonian after a long time development by introducing a proper inner product<sup>\*)</sup> based on the speculation in ref.<sup>17)</sup> If the Hermitian Hamiltonian is given in a local form, a conserved probability current density can be constructed with two kinds of wave functions. We note that the non-Hermitian Hamiltonian studied there is a generic one, so it does not belong to the class of PT symmetric non-Hermitian Hamiltonians, which has recently been intensively studied.<sup>16),18)</sup>

In ref.<sup>14)</sup> introducing a philosophy to keep the analyticity in the parameter variables of FPI and defining a modified set of a complex conjugate, real and imaginary parts, Hermitian conjugates, and bras, we explicitly constructed non-Hermitian operators of coordinate and momentum,  $\hat{q}_{new}$  and  $\hat{p}_{new}$ , and their eigenstates  ${}_m\langle_{new} q|$  and  ${}_m\langle_{new} p|$  for complex  $q$  and  $p$  by utilizing coherent states of harmonic oscillators so that we can deal with complex  $q$  and  $p$ . In addition, applying this complex coordinate formalism to the study of ref.<sup>13)</sup> we showed that the mechanism for suppressing the anti-Hermitian part of the Hamiltonian after a long time development also works in the complex coordinate case. In ref.<sup>15)</sup> based on the complex coordinate formalism we explicitly examined the definitions of the momentum and Hamiltonian via FPI, and confirmed that they have the same forms as those in the real action theory (RAT).

Regarding other studies related to complex coordinates, in refs.<sup>19),20)</sup> the complete set

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\*) A similar inner product was also studied in ref.<sup>16)</sup>

of solutions of the differential equations following from the Schwinger action principle has been obtained by generalizing the path integral to include sums over various inequivalent contours of integration in the complex plane. In ref.,<sup>21)</sup> complex Langevin equations have been studied, and in refs.<sup>22)23)</sup> a method to examine the complexified solution set has been investigated.

In this paper we study a future-included version of the CAT whose path runs over not only past but also future<sup>3)</sup> using both the complex coordinate formalism<sup>14)</sup> and the mechanism for suppressing the anti-Hermitian part of the Hamiltonian.<sup>13)</sup> In ref.,<sup>3)</sup> one of the authors of this paper, H.B.N., and Ninomiya introduced not only the ordinary past state  $|A(T_A)\rangle$  at the initial time  $T_A$ , but also a future state  $|B(T_B)\rangle$  at the final time  $T_B$ , where  $T_A$  and  $T_B$  are set to be  $-\infty$  and  $\infty$  respectively. Here  $|A(T_A)\rangle$  and  $|B(T_B)\rangle$  time-develop according to the non-Hermitian Hamiltonian  $\hat{H}$  and  $\hat{H}_B$ , respectively, where  $\hat{H}_B$  is set to be equal to  $\hat{H}^\dagger$ . They studied the matrix element of some operator  $\mathcal{O}$  defined by

$$\langle \mathcal{O} \rangle^{BA} \equiv \frac{\langle B(t) | \mathcal{O} | A(t) \rangle}{\langle B(t) | A(t) \rangle}, \quad (1.1)$$

where  $t$  is the present time. In the RAT, such a future state as  $|B\rangle$  is already introduced in ref.<sup>24)</sup> in a different context. The matrix element of eq.(1.1), which is called the weak value, has also been intensively studied. For details of the weak value, see the reviews<sup>25)</sup> and references therein. Eq.(1.1) is a matrix element in the usual sense, but in a future-included version of the CAT we speculate that it can be regarded as the expectation value of  $\mathcal{O}$  from the results that we obtain in this paper. As we shall see later,  $\langle \mathcal{O} \rangle^{BA}$  allows us to have the Heisenberg equation. In addition, we shall confirm that it gives us Ehrenfest's theorem. Furthermore, we shall also see that a conserved probability current density can be constructed. Therefore we regard it as an expectation value in the future-included theory.

Here we note that since the future-included theory differs from ordinary quantum mechanics on two points – the existence of the imaginary part of the action  $S$  and that of the future state – it seems excluded phenomenologically. So it is necessary that the future-included theory is not excluded, to show that usual physics is approximately obtained from it. Indeed, in ref.,<sup>3)</sup> an attempt was made to obtain a correspondence between the future-included theory and ordinary quantum mechanics, and it is speculated that  $\langle \mathcal{O} \rangle^{BA}$  becomes

$$\langle \mathcal{O} \rangle^{AA} \equiv \frac{\langle A(t) | \mathcal{O} | A(t) \rangle}{\langle A(t) | A(t) \rangle}, \quad (1.2)$$

i.e. the expectation value of  $\mathcal{O}$  in the future-not-included theory. We review this speculation and make it clear that there are points to be improved in the argument. Then we study  $\langle \mathcal{O} \rangle^{BA}$  with more care concerning the inner product being obtained by using both the complex

coordinate formalism and the mechanism for suppressing the anti-Hermitian part of the Hamiltonian, and show that  $\langle \mathcal{O} \rangle^{BA}$  becomes an expectation value with a different inner product defined in the future-not-included theory. Next we show that the inner product can be interpreted as one of the possible proper inner products realized in the future-not-included theory. Thus we shall have the correspondence principle: the future-included theory for large  $T_B - t$  and large  $t - T_A$  is almost equivalent to the future-not-included theory for large  $t - T_A$ , which means that such theories with complex action and functional integral of future time are not excluded. Incidentally, as for the Hamiltonians in the future-included theory, there are two Hamiltonians  $\hat{H}$  and  $\hat{H}_B$ , but only  $\hat{H}$  is derived in ref.<sup>15)</sup> Therefore, in this paper we give the explicit derivation of  $\hat{H}_B$  via the path integral using the method in ref.,<sup>15)</sup> and confirm that it is given by  $\hat{H}_B = \hat{H}^\dagger$ .

This paper is organized as follows. In section 2 we review our complex coordinate formalism and give a theorem for matrix elements. In section 3 we review the proper inner product for the Hamiltonian  $\hat{H}$ , and introduce another proper inner product for the Hamiltonian  $\hat{H}_B$ . Next we review the mechanism for suppressing the anti-Hermitian part of  $\hat{H}$  after a long time development. In section 4 we study the various properties of the expectation value  $\langle \mathcal{O} \rangle^{BA}$ . We show that it allows us to have the Heisenberg equation, Ehrenfest's theorem and a conserved probability current density. In section 5 after reviewing the study in ref.,<sup>3)</sup> we show that the expectation value of the future-included theory for large  $T_B - t$  and large  $t - T_A$  corresponds to that of the future-not-included theory for large  $t - T_A$  with a proper inner product. Section 6 is devoted to the summary and outlook. In appendix A we give an explicit derivation of  $\hat{H}_B$  via the path integral following ref.<sup>15)</sup>

## §2. Review of the complex coordinate formalism

In this section we briefly review the complex coordinate formalism that we proposed in ref.<sup>14)</sup> so that we can deal with the complex coordinate  $q$  and momentum  $p$  properly in the CAT. We emphasize that even in a real action theory (RAT) we encounter complex  $q$  and  $p$  at the saddle point in the cases of tunneling effect or WKB approximation, etc. As a simple and clear example, let us consider a wave function,

$$\psi(q) = \langle q | \psi \rangle. \quad (2.1)$$

This is defined for real  $q$ , but what happens for complex  $q$  in the cases mentioned above? There are no problems with the left-hand side, because we can just say that the function  $\psi$  is analytically extended to complex  $q$ . But the right-hand side cannot be extended to complex  $q$ , because  $\langle q |$  is defined only for real  $q$ . Indeed  $\langle q |$  obeys  $\langle q | \hat{q} = \langle q | q$ , so if we

attempt to extend the real eigenvalue  $q$  to complex, we encounter a contradiction because  $\hat{q}$  is a Hermitian operator. Therefore  $\hat{q}$  and  $\langle q|$  have to be appropriately extended to a non-Hermitian operator and its eigenstate for complex  $q$ .

2.1. *Non-Hermitian operators  $\hat{q}_{new}$  and  $\hat{p}_{new}$ , and the eigenstates of their Hermitian conjugates  $|q\rangle_{new}$  and  $|p\rangle_{new}$*

Following ref.,<sup>14)</sup> we summarize the construction of the non-Hermitian operators of coordinate and momentum,  $\hat{q}_{new}$  and  $\hat{p}_{new}$ , and the eigenstates of their Hermitian conjugates  $|q\rangle_{new}$  and  $|p\rangle_{new}$  such that

$$\hat{q}_{new}^\dagger |q\rangle_{new} = q |q\rangle_{new}, \quad (2.2)$$

$$\hat{p}_{new}^\dagger |p\rangle_{new} = p |p\rangle_{new}, \quad (2.3)$$

$$[\hat{q}_{new}, \hat{p}_{new}] = i\hbar, \quad (2.4)$$

for complex  $q$  and  $p$  by formally utilizing two coherent states. Our proposal is to replace  $\hat{q}$ ,  $\hat{p}$ ,  $|q\rangle$  and  $|p\rangle$  with  $\hat{q}_{new}^\dagger$ ,  $\hat{p}_{new}^\dagger$ ,  $|q\rangle_{new}$  and  $|p\rangle_{new}$ . The explicit expressions for  $\hat{q}_{new}$ ,  $\hat{p}_{new}$ ,  $|q\rangle_{new}$  and  $|p\rangle_{new}$  are given by

$$\hat{q}_{new} \equiv \frac{1}{\sqrt{1 - \frac{m'\omega'}{m\omega}}} \left( \hat{q} - i \frac{\hat{p}}{m\omega} \right), \quad (2.5)$$

$$\hat{p}_{new} \equiv \frac{1}{\sqrt{1 - \frac{m'\omega'}{m\omega}}} \left( \hat{p} - \frac{m'\omega'}{i} \hat{q} \right), \quad (2.6)$$

$$|q\rangle_{new} \equiv \left\{ \frac{m\omega}{4\pi\hbar} \left( 1 - \frac{m'\omega'}{m\omega} \right) \right\}^{\frac{1}{4}} e^{-\frac{m\omega}{4\hbar} \left( 1 - \frac{m'\omega'}{m\omega} \right) q^2} \left| \sqrt{\frac{m\omega}{2\hbar}} \left( 1 - \frac{m'\omega'}{m\omega} \right) q \right\rangle_{coh}, \quad (2.7)$$

$$|p\rangle_{new} \equiv \left( \frac{1 - \frac{m'\omega'}{m\omega}}{4\pi\hbar m'\omega'} \right)^{\frac{1}{4}} e^{-\frac{1}{4\hbar m'\omega'} \left( 1 - \frac{m'\omega'}{m\omega} \right) p^2} \left| i \sqrt{\frac{1}{2\hbar m'\omega'}} \left( 1 - \frac{m'\omega'}{m\omega} \right) p \right\rangle_{coh}, \quad (2.8)$$

where  $|\lambda\rangle_{coh}$  is a coherent state parametrized with a complex parameter  $\lambda$  defined up to a normalization factor by  $|\lambda\rangle_{coh} \equiv e^{\lambda a^\dagger} |0\rangle = \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle$ , and this satisfies the relation  $a|\lambda\rangle_{coh} = \lambda|\lambda\rangle_{coh}$ . Here  $a = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{q} + i \frac{\hat{p}}{m\omega} \right)$  and  $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{q} - i \frac{\hat{p}}{m\omega} \right)$  are annihilation and creation operators, where  $\hat{q}$  and  $\hat{p}$  are the usual Hermitian operators of coordinate and momentum obeying

$$\hat{q}|q\rangle = q|q\rangle, \quad (2.9)$$

$$\hat{p}|p\rangle = p|p\rangle, \quad (2.10)$$

$$[\hat{q}, \hat{p}] = i\hbar \quad (2.11)$$

for real  $q$  and  $p$ . In eq.(2.8)  $|\lambda\rangle_{coh'}$  is another coherent state, which is defined similarly with different parameters  $m'\omega'$ ,  $|\lambda\rangle_{coh'} \equiv e^{\lambda a'^{\dagger}}|0\rangle$ , where  $a'^{\dagger}$  is given by  $a'^{\dagger} = \sqrt{\frac{m'\omega'}{2\hbar}}(\hat{q} - i\frac{\hat{p}}{m'\omega'})$ . Before seeing the properties of  $\hat{q}_{new}$ ,  $\hat{p}_{new}$ ,  $|q\rangle_{new}$ , and  $|p\rangle_{new}$ , we define a delta function of complex parameters in the next subsection.

## 2.2. The delta function

For our later convenience we first define  $\mathcal{D}$  as a class of distributions depending on one complex variable  $q \in \mathbf{C}$ . Using a function  $g : \mathbf{C} \rightarrow \mathbf{C}$  as a distribution in the class  $\mathcal{D}$ , we define the following functional  $G$

$$G[f] = \int_C f(q)g(q)dq \quad (2.12)$$

for any analytical function  $f : \mathbf{C} \rightarrow \mathbf{C}$  with convergence requirements such that  $f \rightarrow 0$  for  $q \rightarrow \pm\infty$ . The functional  $G$  is a linear mapping from the function  $f$  to a complex number. Since the simulated function  $g$  is supposed to be analytical in  $q$ , the path  $C$ , which is chosen to run from  $-\infty$  to  $\infty$  in the complex plane, can be deformed freely and so it is not relevant. As an approximation to such a distribution we could use the smeared delta function defined for complex  $q$  by

$$g(q) = \delta_c^\epsilon(q) \equiv \sqrt{\frac{1}{4\pi\epsilon}} e^{-\frac{q^2}{4\epsilon}}, \quad (2.13)$$

where  $\epsilon$  is a finite small positive real number. For the limit of  $\epsilon \rightarrow 0$   $g(q)$  converges in the distribution sense for complex  $q$  obeying the condition

$$L(q) \equiv (\text{Re}(q))^2 - (\text{Im}(q))^2 > 0. \quad (2.14)$$

For any analytical test function  $f(q)^*$  and any complex  $q_0$  this  $\delta_c^\epsilon(q)$  satisfies

$$\int_C f(q)\delta_c^\epsilon(q - q_0)dq = f(q_0), \quad (2.15)$$

as long as we choose the path  $C$  such that it runs from  $-\infty$  to  $\infty$  in the complex plane and at any  $q$  its tangent line and a horizontal line form an angle  $\theta$  whose absolute value is within  $\frac{\pi}{4}$  to satisfy the inequality (2.14). An example of permitted paths is shown in fig.1, and the domain of the delta function is drawn in fig.2.

Next we extend the delta function to complex  $\epsilon$ , and consider

$$\delta_c^\epsilon(aq) = \sqrt{\frac{1}{4\pi\epsilon}} e^{-\frac{1}{4\epsilon}a^2q^2} \quad (2.16)$$

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<sup>\*)</sup> Because of the Liouville theorem if  $f$  is a bounded entire function,  $f$  is constant. So we are considering as  $f$  an unbounded entire function or a function that is not entire but is holomorphic at least in the region on which the path runs.

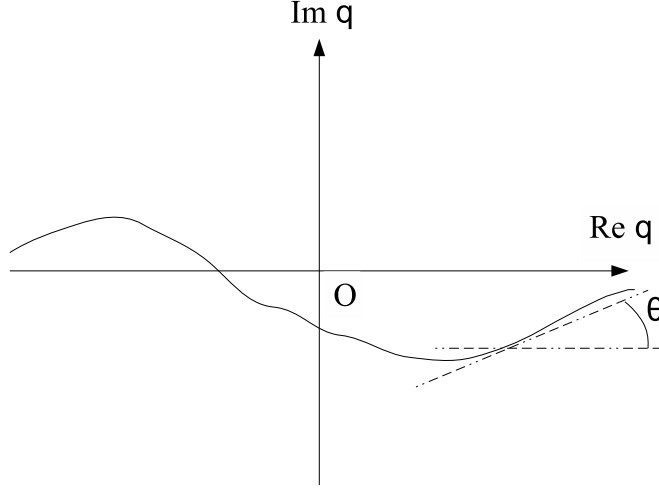


Fig. 1. An example of permitted paths

for non-zero complex  $a$ . We express  $\epsilon$ ,  $q$ , and  $a$  as  $\epsilon = r_\epsilon e^{i\theta_\epsilon}$ ,  $q = r e^{i\theta}$ , and  $a = r_a e^{i\theta_a}$ . The convergence condition of  $\delta_c^\epsilon(aq)$ :  $\text{Re}\left(\frac{a^2 q^2}{\epsilon}\right) > 0$  is expressed as

$$-\frac{\pi}{4} + \frac{1}{2}(\theta_\epsilon - 2\theta_a) < \theta < \frac{\pi}{4} + \frac{1}{2}(\theta_\epsilon - 2\theta_a), \quad (2.17)$$

$$\frac{3}{4}\pi + \frac{1}{2}(\theta_\epsilon - 2\theta_a) < \theta < \frac{5}{4}\pi + \frac{1}{2}(\theta_\epsilon - 2\theta_a). \quad (2.18)$$

For  $q$ ,  $\epsilon$ , and  $a$  such that eqs.(2.17)(2.18) are satisfied,  $\delta_c^\epsilon(aq)$  behaves well as a delta function of  $aq$ , and we obtain the relation

$$\delta_c^\epsilon(aq) = \frac{\text{sign}(\text{Re}a)}{a} \delta_c^{\frac{\epsilon}{a^2}}(q), \quad (2.19)$$

where we have introduced an expression

$$\text{sign}(\text{Re}a) \equiv \begin{cases} 1 & \text{for } \text{Re}(a) > 0, \\ -1 & \text{for } \text{Re}(a) < 0. \end{cases} \quad (2.20)$$

### 2.3. New devices to handle complex parameters

In this subsection, to keep the analyticity in dynamical variables of FPI such as  $q$  and  $p$  we define a modified set of a complex conjugate, Hermitian conjugates, and bras.

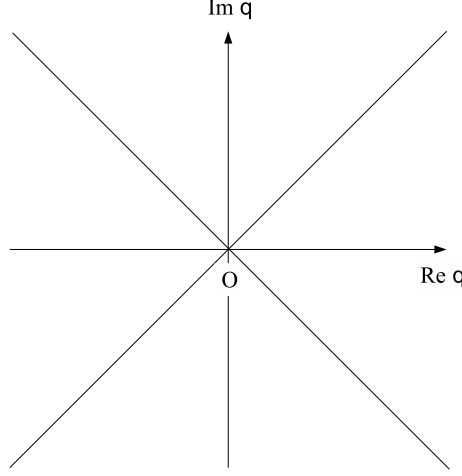


Fig. 2. Domain of the delta function

### 2.3.1. Modified complex conjugate $*_{\{\}}$

We define a modified complex conjugate for a function of  $n$ -parameters  $f(\{a_i\}_{i=1,\dots,n})$  by

$$f(\{a_i\}_{i=1,\dots,n})^{*\{a_i|i \in A\}} = f^*(\{a_i\}_{i \in A}, \{a_i^*\}_{i \notin A}), \quad (2.21)$$

where  $A$  denotes the set of indices attached to the parameters in which we keep the analyticity, and, on the right-hand side,  $*$  on  $f$  acts on the coefficients included in  $f$ . For example, the complex conjugates  $*_q$  and  $*_{q,p}$  of a function  $f(q, p) = aq^2 + bp^2$  are  $f(q, p)^{*a} = a^*q^2 + b^*(p^*)^2$  and  $f(q, p)^{*q,p} = a^*q^2 + b^*p^2$ . The analyticity is kept in  $q$ , and both  $q$  and  $p$ , respectively. For simplicity we express the modified complex conjugate as  $*_{\{\}}$ .

### 2.3.2. Modified bras ${}_m\langle |$ and ${}_{\{\}}\langle |$ , and modified Hermitian conjugate $\dagger_{\{\}}$

For some state  $|\lambda\rangle$  with some complex parameter  $\lambda$ , we define a modified bra  ${}_m\langle \lambda|$  by

$${}_m\langle \lambda| \equiv \langle \lambda^*| \quad (2.22)$$

so that it preserves the analyticity in  $\lambda$ . In the special case of  $\lambda$  being real it becomes a usual bra. In addition we define a slightly generalized modified bra  ${}_{\{\}}\langle |$  and a modified Hermitian conjugate  $\dagger_{\{\}}$  of a ket, where  $\{\}$  is a symbolical expression for a set of parameters in which we keep the analyticity. For example,  ${}_{u,v}\langle u| = {}_u\langle u| = {}_m\langle u|$ ,  $(|u\rangle)^{\dagger_{u,v}} = (|u\rangle)^{\dagger_u} = {}_m\langle u|$ . We



express the Hermitian conjugate  $\dagger_{\{\}}$  of a ket symbolically as  $(| \rangle)^{\dagger_{\{\}}} = \langle |_{\{\}}$ . Also, we write the Hermitian conjugate  $\dagger_{\{\}}$  of a bra as  $(\langle |_{\{\}})^{\dagger_{\{\}}} = | \rangle$ . So for a matrix element we have the relation  $\langle u|A|v\rangle^*_{\{\}} = \langle v|A^\dagger|u\rangle_{\{\}}$ .

#### 2.4. Properties of $\hat{q}_{new}$ , $\hat{p}_{new}$ , $|q\rangle_{new}$ and $|p\rangle_{new}$ , and a theorem for matrix elements

The states  $|q\rangle_{new}$  and  $|p\rangle_{new}$  are normalized so that they satisfy the following relations,

$${}_m\langle_{new} q'|q\rangle_{new} = \delta_c^{\epsilon_1}(q' - q), \quad (2.23)$$

$${}_m\langle_{new} p'|p\rangle_{new} = \delta_c^{\epsilon'_1}(p' - p), \quad (2.24)$$

where  $\epsilon_1 = \frac{\hbar}{m\omega(1-\frac{m'\omega'}{m\omega})}$  and  $\epsilon'_1 = \frac{\hbar m'\omega'}{1-\frac{m'\omega'}{m\omega}}$ . For sufficiently large  $m\omega$  and small  $m'\omega'$  the delta functions converge for complex  $q$ ,  $q'$ ,  $p$ , and  $p'$  satisfying the conditions  $L(q - q') > 0$  and  $L(p - p') > 0$ , where  $L$  is given in eq.(2.14). These conditions are satisfied only when  $q$  and  $q'$  or  $p$  and  $p'$  are on the same paths respectively. In the following we take  $m\omega$  sufficiently large and  $m'\omega'$  sufficiently small. Then eqs.(2.23)(2.24) represent the orthogonality relations for  $|q\rangle_{new}$  and  $|p\rangle_{new}$ , and we have the following relations for complex  $q$  and  $p$ :

$$\int_C dq |q\rangle_{new} {}_m\langle_{new} q| = 1, \quad (2.25)$$

$$\int_C dp |p\rangle_{new} {}_m\langle_{new} p| = 1, \quad (2.26)$$

$$\hat{p}_{new}^\dagger |q\rangle_{new} = i\hbar \frac{\partial}{\partial q} |q\rangle_{new}, \quad (2.27)$$

$$\hat{q}_{new}^\dagger |p\rangle_{new} = \frac{\hbar}{i} \frac{\partial}{\partial p} |p\rangle_{new}, \quad (2.28)$$

$${}_m\langle_{new} q|p\rangle_{new} = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{i}{\hbar}pq\right). \quad (2.29)$$

Thus  $\hat{q}_{new}^\dagger$ ,  $\hat{p}_{new}^\dagger$ ,  $|q\rangle_{new}$  and  $|p\rangle_{new}$  with complex  $q$  and  $p$  obey the same relations that  $\hat{q}$ ,  $\hat{p}$ ,  $|q\rangle$ , and  $|p\rangle$  with real  $q$  and  $p$  satisfy. In the limits of  $m\omega \rightarrow \infty$  and  $m'\omega' \rightarrow 0$   $\delta_c^{\epsilon_1}(q' - q)$ ,  $\delta_c^{\epsilon'_1}(p' - p)$  and  $\exp\left(\frac{i}{\hbar}pq\right)$  in eqs.(2.23)(2.24)(2.29) are well-defined as distributions of the type  $\mathcal{D}$ , which is introduced in subsection 2.2.

For real  $q'$  and  $p'$ ,  $|q'\rangle_{new}$  and  $|p'\rangle_{new}$  become  $|q'\rangle$  and  $|p'\rangle$  respectively. Also, for them,  $\hat{q}_{new}^\dagger$  and  $\hat{p}_{new}^\dagger$  behave like  $\hat{q}$  and  $\hat{p}$  respectively. In relation to the disappearance of the anti-Hermitian terms in  $\hat{q}_{new}^\dagger$  and  $\hat{p}_{new}^\dagger$ , we put forward a theorem for matrix elements of the form  ${}_m\langle_{new} q' \text{ or } p' | \mathcal{O}(\hat{q}_{new}, \hat{q}_{new}^\dagger, \hat{p}_{new}, \hat{p}_{new}^\dagger) | q'' \text{ or } p'' \rangle_{new}$ , where  $\mathcal{O}(\hat{q}_{new}, \hat{q}_{new}^\dagger, \hat{p}_{new}, \hat{p}_{new}^\dagger)$  is a Taylor-expandable function of the four operators  $\hat{q}_{new}$ ,  $\hat{q}_{new}^\dagger$ ,  $\hat{p}_{new}$ , and  $\hat{p}_{new}^\dagger$ . We easily see that such a matrix element can be expressed as the summation of the products of factors made of  $q'$ ,  $p'$ ,  $q''$ ,  $p''$  or their differential operators and the distributions  $\delta_c^{\epsilon_1}(q' - q'')$ ,  $\delta_c^{\epsilon'_1}(p' - p'')$ ,

or  $\exp(\pm \frac{i}{\hbar} p' q'')$ . Then, since we shall extract only analytically weighted results from the matrix element, we do not have to worry about the anti-hermitian terms in  $\hat{q}_{new}$ ,  $\hat{q}_{new}^\dagger$ ,  $\hat{p}_{new}$  and  $\hat{p}_{new}^\dagger$ , provided that we are satisfied with the result in the distribution sense. So we pose the following theorem.

Theorem: The matrix element  ${}_m \langle_{new} q' \text{ or } p' | \mathcal{O}(\hat{q}_{new}, \hat{q}_{new}^\dagger, \hat{p}_{new}, \hat{p}_{new}^\dagger) | q'' \text{ or } p'' \rangle_{new}$  can be evaluated as if inside the operator  $\mathcal{O}$  we had the hermiticity conditions  $\hat{q}_{new} \simeq \hat{q}_{new}^\dagger \simeq \hat{q}$  and  $\hat{p}_{new} \simeq \hat{p}_{new}^\dagger \simeq \hat{p}$  for  $q'$ ,  $q''$ ,  $p'$ ,  $p''$  such that the resulting quantities are well-defined in the sense of distribution.

This theorem could help us from running into calculations that are too hard for the use of our complex coordinate formalism.

### 2.5. Remarks on the complex coordinate formalism

We have seen that  $\hat{q}_{new}$  and  $\hat{p}_{new}$  etc. have nice properties, but one might still feel a bit uneasy about our replacement of  $\hat{q}$  and  $\hat{p}$  with  $\hat{q}_{new}$  and  $\hat{p}_{new}$ . To accept  $\hat{q}_{new}$  and  $\hat{p}_{new}$ , it might help slightly to have in mind that operators smooth in  $\hat{q}$  and  $\hat{p}$  like  $\hat{q}_{new}$  and  $\hat{p}_{new}$  generically have eigenvalues filling the whole complex plane, while Hermitian operators like  $\hat{q}$  and  $\hat{p}$  have eigenvalues only along a certain curve, e.g., on the real axis in the complex plane. For our purpose of having general contours running through eigenvalues we replaced the special operators  $\hat{q}$  and  $\hat{p}$  by the more generic ones  $\hat{q}_{new}$  and  $\hat{p}_{new}$ . The philosophy should be that almost any small disturbance would anyway bring  $\hat{q}$  and  $\hat{p}$  into operators of the generic type with the whole complex plane as a spectrum. The operators  $\hat{q}_{new}$  and  $\hat{p}_{new}$  are just concrete examples of such tiny deformation. So we stress that the Hermitian operators as  $\hat{q}$  and  $\hat{p}$  are special by having their eigenvalue spectrum on a curve, e.g., on the real axis in the complex plane rather than distributed all over it. If we had clung to the belief in curve-spectra, it would have been embarrassing for our formalism that under Heisenberg time development one could have feared that, from time to time in our scheme, the curve-spectra would be transformed into new curve-spectra that might not match at the free contour choice. Now, however, as already stressed, if we use  $\hat{q}_{new}$  and  $\hat{p}_{new}$ , from the beginning we have already gone over to operators with any complex numbers as eigenvalues. So arbitrary deformation of the contour would a priori cause no problems. Thus we claim that the contours of integration can be chosen freely at each time  $t$ , so that there is no need for any natural choice, which only has to run from  $-\infty$  to  $\infty$ .

We come back to the problem we raised at the beginning of this section: how eq.(2.1) is expressed for complex  $q$ . Now we can express it based on our complex coordinate formalism

as

$$\psi(q) = {}_m\langle_{new} q|\psi\rangle. \quad (2.30)$$

When  $q$  is real, this becomes the usual expression of eq.(2.1). In addition we mention another problem on the momentum relation  $p = m\dot{q}$  in the CAT, where the mass  $m$  is generically complex. Indeed we encounter a contradiction if we cling to the real eigenvalues  $q$  and  $p$  of the usual Hermitian operators  $\hat{q}$  and  $\hat{p}$ . In ref.,<sup>15)</sup> we have explicitly examined the definitions of the momentum and Hamiltonian via FPI, and confirmed that they have the same forms as those in the RAT but with complex eigenvalues  $q$  and  $p$  of non-Hermitian operators  $\hat{q}_{new}$  and  $\hat{p}_{new}$ . In this section, we have briefly reviewed the complex coordinate formalism; for details see ref.<sup>14)</sup> Finally, we show a summary of the comparison between the RAT and the CAT in Table I.

Table I. Various quantities in the RAT and the CAT

	the RAT	the CAT
parameters	$q, p$ real,	$q, p$ complex
complex conjugate	*	*{ }
Hermitian conjugate	†	†{ }
delta function of $q$	$\delta(q)$ defined for real $q$	$\delta_c(q)$ defined for $q$ s.t. $(\text{Re}(q))^2 > (\text{Im}(q))^2$
bras of $ q\rangle,  p\rangle$	$\langle q  = ( q\rangle)^\dagger,$ $\langle p  = ( p\rangle)^\dagger$	${}_m\langle_{new} q  = \langle_{new} q^*  = ( q\rangle_{new})^{\dagger_q},$ ${}_m\langle_{new} p  = \langle_{new} p^*  = ( p\rangle_{new})^{\dagger_p}$
completeness for $ q\rangle$ and $ p\rangle$	$\int_{-\infty}^{\infty}  q\rangle\langle q dq = 1,$ $\int_{-\infty}^{\infty}  p\rangle\langle p dp = 1$ along real axis	$\int_C  q\rangle_{new} {}_m\langle_{new} q dq = 1,$ $\int_C  p\rangle_{new} {}_m\langle_{new} p dp = 1$ $C$ : any path running from $-\infty$ to $\infty$
orthogonality for $ q\rangle$ and $ p\rangle$	$\langle q q'\rangle = \delta(q - q'),$ $\langle p p'\rangle = \delta(p - p')$	${}_m\langle_{new} q q'\rangle_{new} = \delta_c^{\epsilon_1}(q - q'),$ ${}_m\langle_{new} p p'\rangle_{new} = \delta_c^{\epsilon'_1}(p - p')$
basis of Fourier expansion	$\langle q p\rangle = \exp(ipq)$	${}_m\langle_{new} q p\rangle_{new} = \exp(ipq)$
$q$ representation of $ \psi\rangle$	$\psi(q) = \langle q \psi\rangle$	$\psi(q) = {}_m\langle_{new} q \psi\rangle$
complex conjugate of $\psi(q)$	$\langle q \psi\rangle^* = \langle\psi q\rangle$	${}_m\langle_{new} q \psi\rangle^{*q} = \langle\psi q\rangle_{new}$
normalization of $\psi(q)$	$\int_{-\infty}^{\infty} \psi(q)^*\psi(q)dq = 1$	$\int_C \psi(q)^{*q}\psi(q)dq = 1$

### §3. Proper inner products and the automatic hermiticity mechanism

We begin with an explanation of the two states  $\langle B(t)|$  and  $|A(t)\rangle$ , definable from their respective functional integrals of future and past following ref.<sup>3)</sup> and review the proper inner product for the Hamiltonian  $\hat{H}$  in the future-not-included theory following ref.<sup>13),14)</sup> Then we construct the proper inner product for the other Hamiltonian  $\hat{H}_B$  in the future-included theory. Furthermore we review the mechanism<sup>13),14)</sup> for suppressing the anti-Hermitian part of the Hamiltonian.

#### 3.1. Definitions of $|A(t)\rangle$ and $\langle B(t)|$

In ref.<sup>3)</sup> the state  $|A(t)\rangle$  is introduced by  $\langle q|A(t)\rangle = \int_{\text{path}(t)=q} e^{\frac{i}{\hbar}S_{-\infty \text{ to } t}} D\text{path}$ , where  $\text{path}(t) = q$  means the boundary condition at the time  $t$ . We write this in a slightly modified form as

$$\psi_A(q) = {}_m\langle_{\text{new}} q|A(t)\rangle = \int_{\text{path}(t)=q} e^{\frac{i}{\hbar}S_{T_A \text{ to } t}} D\text{path} \quad (3.1)$$

based on the complex coordinate formalism<sup>14)</sup> so that it is properly defined even for complex  $q$ . The other state  $|B(t)\rangle$  is introduced in ref.<sup>3)</sup> as  $\langle B(t)|q\rangle \equiv \int_{\text{path}(t)=q} e^{\frac{i}{\hbar}S_t \text{ to } \infty} D\text{path}$ , but we express it as

$$\psi_B(q)^{*q} = \langle B(t)|q\rangle_{\text{new}} = \int_{\text{path}(t)=q} e^{\frac{i}{\hbar}S_t \text{ to } \infty} D\text{path}, \quad (3.2)$$

by defining  $\psi_B(q)$  by

$$\psi_B(q) = {}_m\langle_{\text{new}} q|B\rangle. \quad (3.3)$$

Here  $\psi_A(q)$  and  $\psi_B(q)$  are kinds of wave functions of the universe at time  $t$ , which include information on the past and future times, respectively. The states  $|A(t)\rangle$  and  $|B(t)\rangle$  time-develop according to

$$i\hbar \frac{d}{dt}|A(t)\rangle = \hat{H}|A(t)\rangle, \quad (3.4)$$

$$i\hbar \frac{d}{dt}|B(t)\rangle = \hat{H}_B|B(t)\rangle, \quad (3.5)$$

where

$$\hat{H}_B = \hat{H}^\dagger. \quad (3.6)$$

We note that in ref.<sup>15)</sup> we explicitly derived the form of  $\hat{H}$  – in a system with a single degree of freedom for simplicity – via the Feynman path integral as follows:

$$\hat{H} = \frac{1}{2m}(\hat{p}_{\text{new}})^2 + V(\hat{q}_{\text{new}}), \quad (3.7)$$

where we supposed that  $V(\hat{q}_{\text{new}}) = \sum_{n=2}^{\infty} b_n(\hat{q}_{\text{new}})^n$ . This is the first application of the complex coordinate formalism. In appendix A we derive  $\hat{H}_B$  in a similar way.

### 3.2. A proper inner product for $\hat{H}$

As preparation for deriving the proper inner product for  $\hat{H}_B$  we briefly review and explain the construction of the proper inner product for  $\hat{H}$ , which we introduced in refs.<sup>13),14)</sup> in a future-not-included theory. To explain it we introduce the eigenstates  $|\lambda_i\rangle (i = 1, 2, \dots)$  of the Hamiltonian  $\hat{H}$  obeying  $\hat{H}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$ , where  $\lambda_i (i = 1, 2, \dots)$  are the eigenvalues of  $\hat{H}$ , and define the diagonalizing operator  $P$  of the Hamiltonian  $\hat{H}$  by  $P = (|\lambda_1\rangle, |\lambda_2\rangle, \dots)$ . Then  $\hat{H}$  is diagonalized as  $\hat{H} = PDP^{-1}$ , where  $D$  is given by  $\text{diag}(\lambda_1, \lambda_2, \dots)$ . We introduce an orthonormal basis  $|e_i\rangle (i = 1, \dots)$  satisfying  $\langle e_i | e_j \rangle = \delta_{ij}$  by  $D|e_i\rangle = \lambda_i|e_i\rangle$ . The basis  $|e_i\rangle$  is related to  $|\lambda_i\rangle$  as  $|\lambda_i\rangle = P|e_i\rangle$ . We note that the  $|\lambda_i\rangle$  are not orthogonal to each other in the usual inner product  $I$ ,  $I(|\lambda_i\rangle, |\lambda_j\rangle) = \langle \lambda_i | \lambda_j \rangle \neq \delta_{ij}$ . So the theory defined with  $I$  would measure unphysical transitions. To make a physically reasonable measurement, in refs.<sup>13),14)</sup> we introduced a proper inner product  $I_Q$  for arbitrary kets  $|u\rangle$  and  $|v\rangle$  as

$$I_Q(|u\rangle, |v\rangle) = \langle u |_Q v \rangle = \langle u | Q | v \rangle, \quad (3.8)$$

where  $Q$  is a Hermitian operator chosen as

$$Q = (P^\dagger)^{-1} P^{-1} \quad (3.9)$$

so that the eigenstates of  $\hat{H}$  become orthogonal to each other with regard to  $I_Q$ ,  $I_Q(|\lambda_i\rangle, |\lambda_j\rangle) = \delta_{ij}$ . This enables us to make a physically reasonable observation, and also implies the orthogonality relation  $\sum_i |\lambda_i\rangle \langle \lambda_i|_Q = 1$ . We note that  $I_Q$  is different from the CPT inner product defined in the PT symmetric Hamiltonian formalism.<sup>18)</sup>

Via the inner product  $I_Q$  we define the  $Q$ -Hermitian conjugate of some operator  $A$  by  $\langle \psi_2 |_Q A | \psi_1 \rangle^* = \langle \psi_1 |_Q A^{\dagger Q} | \psi_2 \rangle$ , from which we see that  $A^{\dagger Q}$  is written as  $A^{\dagger Q} = Q^{-1} A^\dagger Q$ . Similarly, in the case where  $|\psi_1\rangle$  or  $|\psi_2\rangle$  are given in states as  $|u\rangle$  or  $|v\rangle$ , we can consider the following relation:  $\{ \langle v |_Q A | u \rangle^* \} = \{ \langle u |_Q A^{\dagger Q} | v \rangle \}$ , where  $\{ \}$  denotes a set of parameters in which we keep the analyticity. We also define  $\dagger^Q$  for kets and bras as  $|\lambda\rangle^{\dagger Q} \equiv \langle \lambda |_Q$  and  $(\langle \lambda |_Q)^{\dagger Q} \equiv |\lambda\rangle$ . Similarly, we define  $\dagger_{\{ \}}^Q$  for kets and bras as  $|\lambda\rangle^{\dagger_{\{ \}}^Q} \equiv \{ \langle \lambda |_Q$  and  $(\{ \langle \lambda |_Q)^{\dagger_{\{ \}}^Q} \equiv |\lambda\rangle$ . When some operator  $A$  satisfies  $A^{\dagger Q} = A$ , we call  $A$   $Q$ -Hermitian. This is the definition of  $Q$ -hermiticity. \*)

Furthermore we explain the  $Q$ -normality of  $H$ . Since

$$“P^{\dagger Q}” \equiv \begin{pmatrix} \langle \lambda_1 |_Q \\ \langle \lambda_2 |_Q \\ \vdots \end{pmatrix} = P^{-1} \quad (3.10)$$

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\*) We note that in ref.<sup>16)</sup> a similar inner product was studied and a criterion for identifying a unique inner product through the choice of physical observables was also provided.

satisfies “ $P^{\dagger Q}$ ”  $\hat{H}P = D$  and “ $P^{\dagger Q}$ ”  $\hat{H}^{\dagger Q}P = D^\dagger$ ,  $\hat{H}$  is  $Q$ -normal,  $[\hat{H}, \hat{H}^{\dagger Q}] = P[D, D^\dagger]P^{-1} = 0$ . In other words the inner product  $I_Q$  is defined so that  $\hat{H}$  is normal with regard to it.

### 3.3. A proper inner product for $\hat{H}_B$

Following the construction of the proper inner product for  $\hat{H}$  in the previous subsection we construct the proper inner product  $I_{Q_B}$  for  $\hat{H}_B$ . Taking the Hermitian conjugate of the relation  $\langle \lambda_j |_Q \hat{H} = \lambda_j \langle \lambda_j |_Q$ , we obtain  $\hat{H}^\dagger Q |\lambda_j\rangle = \lambda_j^* Q |\lambda_j\rangle$ . Using eq.(3.6) we rewrite this as

$$\hat{H}_B |\lambda_j\rangle_B = \lambda_j^* |\lambda_j\rangle_B, \quad (3.11)$$

where we have introduced  $|\lambda_j\rangle_B \equiv Q |\lambda_j\rangle$ . Thus the eigenstates and the eigenvalues of  $\hat{H}_B$  are given by  $|\lambda_j\rangle_B$  and  $\lambda_j^*$  ( $j = 1, 2, \dots$ ) respectively, and the diagonalizing matrix of  $\hat{H}_B$  is given by  $P_B \equiv (|\lambda_1\rangle_B, |\lambda_2\rangle_B, \dots) = QP = (P^\dagger)^{-1}$ . We introduce a proper inner product  $I_{Q_B}$  for arbitrary kets  $|u\rangle$  and  $|v\rangle$  as  $I_{Q_B}(|u\rangle, |v\rangle) = \langle u |_{Q_B} v \rangle = \langle u | Q_B | v \rangle$ , where  $Q_B$  is a Hermitian operator chosen as

$$Q_B = (P_B^\dagger)^{-1} P_B^{-1} = Q^{-1} \quad (3.12)$$

in order that  $|\lambda_j\rangle_B$  become orthogonal to each other with regard to  $I_{Q_B}$ ,  $I_{Q_B}(|\lambda_i\rangle_B, |\lambda_j\rangle_B) = \delta_{ij}$ . Then we also have the completeness relation  $\sum_i |\lambda_i\rangle_B {}_B\langle \lambda_i|_{Q_B} = 1$ .

Taking the  $Q_B$ -Hermitian conjugate of eq.(3.11), we obtain  ${}_B\langle \lambda_i |_{Q_B} \hat{H}_B^{\dagger Q_B} = {}_B\langle \lambda_i |_{Q_B} \lambda_i$ , where  $\hat{H}_B^{\dagger Q_B}$  is given by

$$\hat{H}_B^{\dagger Q_B} = Q_B^{-1} \hat{H}_B^\dagger Q_B. \quad (3.13)$$

Since

$$“P_B^{\dagger Q_B}” \equiv \begin{pmatrix} {}_B\langle \lambda_1 |_{Q_B} \\ {}_B\langle \lambda_2 |_{Q_B} \\ \vdots \end{pmatrix} = (P_B)^{-1} \quad (3.14)$$

satisfies “ $P_B^{\dagger Q_B}$ ”  $\hat{H}_B P_B = D^\dagger$  and “ $P_B^{\dagger Q_B}$ ”  $\hat{H}_B^{\dagger Q_B} P_B = D$ ,  $\hat{H}_B$  is  $Q_B$ -normal,  $[\hat{H}_B, \hat{H}_B^{\dagger Q_B}] = 0$ .

For later convenience we decompose  $\hat{H}$  as  $\hat{H} = \hat{H}_{Q_h} + \hat{H}_{Q_a}$ , where  $\hat{H}_{Q_h} = \frac{\hat{H} + \hat{H}^{\dagger Q}}{2}$  and  $\hat{H}_{Q_a} = \frac{\hat{H} - \hat{H}^{\dagger Q}}{2}$  are  $Q$ -Hermitian and anti- $Q$ -Hermitian parts of  $\hat{H}$  respectively. We also decompose  $D$  as  $D = D_R + iD_I$ , where we have introduced  $D_R = \frac{D + D^\dagger}{2}$  and  $D_I = \frac{D - D^\dagger}{2}$ . The diagonal components of  $D_R$  and  $D_I$  are the real and imaginary parts of the diagonal components of  $D$  respectively. Then  $\hat{H}_{Q_h}$  and  $\hat{H}_{Q_a}$  can be expressed in terms of  $D_R$  and  $D_I$  as  $\hat{H}_{Q_h} = PD_R P^{-1}$  and  $\hat{H}_{Q_a} = iPD_I P^{-1}$ .

### 3.4. The automatic hermiticity mechanism

In this subsection we give a brief review of the mechanism for suppressing the effect of  $\hat{H}_{Q_a}$  after a long time development of some state  $|\psi(t)\rangle$ , which obeys the Schrödinger equation

$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$ , by following refs.<sup>13),14)</sup> We introduce  $|\psi'(t)\rangle$  by  $|\psi'(t)\rangle = P^{-1}|\psi(t)\rangle$ , and expand it as  $|\psi'(t)\rangle = \sum_i a_i(t)|e_i\rangle$ . Then  $|\psi(t)\rangle$  can be written in an expanded form as  $|\psi(t)\rangle = \sum_i a_i(t)|\lambda_i\rangle$ . Since  $|\psi'(t)\rangle$  obeys  $i\hbar \frac{d}{dt}|\psi'(t)\rangle = D|\psi'(t)\rangle$ , the time development of  $|\psi(t)\rangle$  from some time  $t_0$  is calculated as

$$\begin{aligned} |\psi(t)\rangle &= P e^{-\frac{i}{\hbar} D(t-t_0)} |\psi'(t_0)\rangle \\ &= \sum_i a_i(t_0) e^{\frac{1}{\hbar} (\text{Im}\lambda_i - i\text{Re}\lambda_i)(t-t_0)} |\lambda_i\rangle. \end{aligned} \quad (3.15)$$

Now we assume that the anti-Hermitian part of  $\hat{H}$  is bounded from above. We point out that this is a natural assumption in our CAT because this allows the whole FPI  $= \int e^{\frac{i}{\hbar} S} \mathcal{D}\text{path}$  to converge. Indeed, this integral diverges unless the imaginary part of the action  $S_I$  is bounded from below. In ref.<sup>15)</sup> to prevent the kinetic term from blowing up for  $\dot{q} \rightarrow \pm\infty$ , we gave a condition  $\text{Im}(m) \geq 0$  on the mass  $m$ , which is equivalent to  $\text{Im}(\frac{1}{m}) \leq 0$ . In addition the imaginary part of the potential term  $\text{Im}(V)$  has to be bounded from above. Thus the assumption of the boundedness of  $H$  is needed to avoid the FPI  $= \int e^{\frac{i}{\hbar} S} \mathcal{D}\text{path}$  being divergently meaningless.

Based on this assumption we can crudely imagine that some of the  $\text{Im}\lambda_i$  take the maximal value  $B$ . We denote the corresponding subset of  $\{i\}$  as  $A$ . Then, if a long time has passed, namely for large  $t - t_0$ , the states with  $\text{Im}\lambda_i|_{i \in A}$  survive and contribute most in the sum. To show how  $|\psi(t)\rangle$  is effectively described for large  $t - t_0$ , we introduce a diagonalized Hamiltonian  $\tilde{D}_R$  as

$$\langle e_i | \tilde{D}_R | e_j \rangle \equiv \begin{cases} \langle e_i | D_R | e_j \rangle = \delta_{ij} \text{Re}\lambda_i & \text{for } i \in A, \\ 0 & \text{for } i \notin A, \end{cases} \quad (3.16)$$

and define  $\hat{H}_{\text{eff}}$  by  $\hat{H}_{\text{eff}} \equiv P \tilde{D}_R P^{-1}$ . Since  $(\tilde{D}_R)^\dagger = \tilde{D}_R$ ,  $\hat{H}_{\text{eff}}$  is  $Q$ -Hermitian,  $\hat{H}_{\text{eff}}^{\dagger Q} = \hat{H}_{\text{eff}}$ , and satisfies  $\hat{H}_{\text{eff}}|\lambda_i\rangle = \text{Re}\lambda_i|\lambda_i\rangle$ . Furthermore, we introduce  $|\tilde{\psi}(t)\rangle \equiv \sum_{i \in A} a_i(t)|\lambda_i\rangle$ . Then  $|\psi(t)\rangle$  is approximately estimated as

$$\begin{aligned} |\psi(t)\rangle &\simeq e^{\frac{1}{\hbar} B(t-t_0)} \sum_{i \in A} a_i(t_0) e^{-\frac{i}{\hbar} \text{Re}\lambda_i(t-t_0)} |\lambda_i\rangle \\ &= e^{\frac{1}{\hbar} B(t-t_0)} e^{-\frac{i}{\hbar} \hat{H}_{\text{eff}}(t-t_0)} |\tilde{\psi}(t_0)\rangle = |\tilde{\psi}(t)\rangle. \end{aligned} \quad (3.17)$$

The factor  $e^{\frac{1}{\hbar} B(t-t_0)}$  in eq.(3.17) can be dropped out by normalization. Thus we have effectively obtained a  $Q$ -Hermitian Hamiltonian  $\hat{H}_{\text{eff}}$  after a long time development.

Indeed the normalized state

$$|\psi(t)\rangle_N \equiv \frac{1}{\sqrt{\langle \psi(t) |_Q \psi(t) \rangle}} |\psi(t)\rangle \simeq \frac{1}{\sqrt{\langle \tilde{\psi}(t) |_Q \tilde{\psi}(t) \rangle}} |\tilde{\psi}(t)\rangle \equiv |\tilde{\psi}(t)\rangle_N \quad (3.18)$$

originally obeys the slightly modified Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle_N = \hat{H}_{Qh} |\psi(t)\rangle_N + \left( \hat{H}_{Qa} - {}_N\langle\psi(t)|_Q \hat{H}_{Qa} |\psi(t)\rangle_N \right) |\psi(t)\rangle_N, \quad (3.19)$$

but after a long time it time-develops as  $|\tilde{\psi}(t)\rangle_N = e^{-\frac{i}{\hbar} \hat{H}_{\text{eff}}(t-t_0)} |\tilde{\psi}(t_0)\rangle_N$ , i.e. it obeys the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\tilde{\psi}(t)\rangle_N = \hat{H}_{\text{eff}} |\tilde{\psi}(t)\rangle_N. \quad (3.20)$$

We see that the time dependence of the normalization factor has disappeared.

On the other hand, we define the expectation value of some operator  $\mathcal{O}$  by

$$\begin{aligned} \bar{\mathcal{O}}_Q(t) &\equiv {}_N\langle\psi(t)|_Q \mathcal{O} |\psi(t)\rangle_N = {}_N\langle\psi(t_0)|_Q \mathcal{O}_{QH}(t-t_0) |\psi(t_0)\rangle_N \\ &\simeq {}_N\langle\tilde{\psi}(t)|_Q \mathcal{O} |\tilde{\psi}(t)\rangle_N = {}_N\langle\tilde{\psi}(t_0)|_Q \tilde{\mathcal{O}}_{QH}(t-t_0) |\tilde{\psi}(t_0)\rangle_N, \end{aligned} \quad (3.21)$$

where we have introduced the time-dependent operator in the Heisenberg picture,

$$\begin{aligned} \mathcal{O}_{QH}(t-t_0) &\equiv \frac{\langle\psi(t_0)|_Q \psi(t_0)\rangle}{\langle\psi(t)|_Q \psi(t)\rangle} e^{\frac{i}{\hbar} \hat{H}^{\dagger Q}(t-t_0)} \mathcal{O} e^{-\frac{i}{\hbar} \hat{H}(t-t_0)} \\ &\simeq e^{\frac{i}{\hbar} \hat{H}_{\text{eff}}(t-t_0)} \mathcal{O} e^{-\frac{i}{\hbar} \hat{H}_{\text{eff}}(t-t_0)} \equiv \tilde{\mathcal{O}}_{QH}(t-t_0). \end{aligned} \quad (3.22)$$

The time-dependent operator originally obeys the slightly modified Heisenberg equation,

$$\begin{aligned} &i\hbar \frac{\partial}{\partial t} \mathcal{O}_{QH}(t-t_0) \\ &= [\mathcal{O}_{QH}(t-t_0), \hat{H}_{Qh}] + \left\{ \mathcal{O}_{QH}(t-t_0), \hat{H}_{Qa} - {}_N\langle\psi(t)|_Q \hat{H}_{Qa} |\psi(t)\rangle_N \right\}, \end{aligned} \quad (3.23)$$

but after a long time development it obeys the Heisenberg equation

$$i\hbar \frac{\partial}{\partial t} \tilde{\mathcal{O}}_{QH}(t-t_0) = \frac{i}{\hbar} [\hat{H}_{\text{eff}}, \tilde{\mathcal{O}}_{QH}(t-t_0)]. \quad (3.24)$$

As we have seen above, the non-Hermitian Hamiltonian  $\hat{H}$  has automatically become a hermitian one  $\hat{H}_{\text{eff}}$  with the proper inner product  $I_Q$  and a long time development.

#### §4. Nice properties of the expectation value $\langle\mathcal{O}\rangle^{BA}$

In a future-included version of the CAT,  $\langle\mathcal{O}\rangle^{BA}$ , defined in eq.(1.1), was considered as an expectation value of  $\mathcal{O}$  in ref.,<sup>3)</sup> although this is a matrix element in the usual sense. A similar form was also considered in ref.<sup>24)</sup> in a different context. In this section we study this quantity further in the CAT and explicitly show that  $\langle\mathcal{O}\rangle^{BA}$  has nice properties: it allows us to have the Heisenberg equation, Ehrenfest's theorem, and a conserved probability current density. These properties strongly suggest that  $\langle\mathcal{O}\rangle^{BA}$  is a promising definition of the expectation value in the future-included CAT.



#### 4.1. Heisenberg equation

From eqs.(3.4)(3.5),  $|A(t)\rangle$  and  $|B(t)\rangle$  time-develop as  $|A(t)\rangle = \exp\left(-\frac{i}{\hbar}\hat{H}(t - T_A)\right)|A(T_A)\rangle$  and  $|B(t)\rangle = \exp\left(-\frac{i}{\hbar}\hat{H}(t - T_B)\right)|B(T_B)\rangle$ , where we have supposed a future final state  $|B(T_B)\rangle$  and a past initial state  $|A(T_A)\rangle$ . As for eq.(1.1), we note that the denominator  $\langle B(t)|A(t)\rangle$  is constant in time,  $\frac{d}{dt}\langle B(t)|A(t)\rangle = 0$ . We attempt to write the numerator as  $\langle B(t)|\mathcal{O}|A(t)\rangle = \langle B(T_B)|\hat{\mathcal{O}}_H(t)|A(T_A)\rangle$ , where we have defined a Heisenberg operator

$$\hat{\mathcal{O}}_H(t) \equiv \exp\left(\frac{i}{\hbar}\hat{H}(t - T_B)\right)\mathcal{O}\exp\left(-\frac{i}{\hbar}\hat{H}(t - T_A)\right) \quad (4.1)$$

obeying the Heisenberg equation  $\frac{d}{dt}\hat{\mathcal{O}}_H(t) = \frac{i}{\hbar}[H, \hat{\mathcal{O}}_H(t)]$ . But we encounter

$$\hat{1}_H(t) = e^{\frac{i}{\hbar}\hat{H}(t_0 - T_B)}, \quad (4.2)$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle^{BA} = \frac{\langle B(T_B)|\hat{\mathcal{O}}_{1H}(t)e^{\frac{i}{\hbar}\hat{H}(T_B - T_A)}\hat{\mathcal{O}}_{2H}(t)|A(T_A)\rangle}{\langle B(T_B)|e^{-\frac{i}{\hbar}\hat{H}(T_B - T_A)}|A(T_A)\rangle}, \quad (4.3)$$

which are not usual expressions. To avoid this situation we rewrite the numerator of eq.(1.1) with some reference time  $t_{\text{ref}}$ , which can be chosen arbitrarily such that  $T_A \leq t_{\text{ref}} \leq T_B$ , as  $\langle B(t)|\mathcal{O}|A(t)\rangle = \langle B(t_{\text{ref}})|\hat{\mathcal{O}}_H(t; t_{\text{ref}})|A(t_{\text{ref}})\rangle$ , where we have defined another Heisenberg operator

$$\hat{\mathcal{O}}_H(t; t_{\text{ref}}) \equiv \exp\left(\frac{i}{\hbar}\hat{H}(t - t_{\text{ref}})\right)\mathcal{O}\exp\left(-\frac{i}{\hbar}\hat{H}(t - t_{\text{ref}})\right) \quad (4.4)$$

obeying the Heisenberg equation  $\frac{d}{dt}\hat{\mathcal{O}}_H(t; t_{\text{ref}}) = \frac{i}{\hbar}[\hat{H}, \hat{\mathcal{O}}_H(t; t_{\text{ref}})]$ . In contrast to eqs.(4.2)(4.3), we have the following usual expressions:

$$\hat{1}_H(t; t_{\text{ref}}) = 1, \quad (4.5)$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle^{BA} = \frac{\langle B(t_{\text{ref}})|\hat{\mathcal{O}}_{1H}(t; t_{\text{ref}})\hat{\mathcal{O}}_{2H}(t; t_{\text{ref}})|A(t_{\text{ref}})\rangle}{\langle B(t_{\text{ref}})|A(t_{\text{ref}})\rangle}. \quad (4.6)$$

So we adopt the expression (4.4) for the Heisenberg operator in our theory.

Before finishing this subsection we make a remark on  $\langle \mathcal{O} \rangle^{BA}$  and the contour  $C$  in the path integral. The expectation value  $\langle \mathcal{O} \rangle^{BA}$  is not real even for Hermitian  $\mathcal{O}$ , and would usually become so complicated that it would typically have values all over the complex plane  $\mathbf{C}$ . This is in contrast to the expectation value  $\langle \mathcal{O} \rangle^{AA}$  in the future-not-included theory, which is real for Hermitian  $\mathcal{O}$ . For both  $\langle \mathcal{O} \rangle^{BA}$  and  $\langle \mathcal{O} \rangle^{AA}$  there is no problem in deforming the integration contour  $C$  at each time arbitrarily.

#### 4.2. Ehrenfest's theorem

In this subsection we derive Ehrenfest's theorem. Utilizing the following relation

$$\frac{d}{dt}\langle \mathcal{O} \rangle^{BA} = \langle \frac{i}{\hbar}[\hat{H}, \mathcal{O}] \rangle^{BA}, \quad (4.7)$$

where we use  $\hat{H}$  given in eq.(3.7), we obtain

$$\frac{d}{dt}\langle\hat{q}_{new}\rangle^{BA} = \frac{1}{m}\langle\hat{p}_{new}\rangle^{BA}, \quad (4.8)$$

$$\frac{d}{dt}\langle\hat{p}_{new}\rangle^{BA} = -\langle V'(\hat{q}_{new})\rangle^{BA}. \quad (4.9)$$

We note that eq.(4.8) is consistent with the momentum relation derived via the path integral in ref.<sup>15)</sup> Substituting eq.(4.8) for eq.(4.9), we obtain Ehrenfest's theorem,

$$m\frac{d^2}{dt^2}\langle\hat{q}_{new}\rangle^{BA} = -\langle V'(\hat{q}_{new})\rangle^{BA}. \quad (4.10)$$

We have thus checked that  $\langle\mathcal{O}\rangle^{BA}$  provides the saddle point development with  $t$ .

#### 4.3. Conserved probability current density

In this subsection we show that a conserved probability current density can be constructed in the future-included theory. First we define a probability density  $\rho$  by

$$\rho \equiv \frac{\psi_B(q)^{*q}\psi_A(q)}{\langle B|A\rangle} = \frac{\langle B|q\rangle_{new} \frac{1}{m}\langle_{new} q|A\rangle}{\langle B|A\rangle}, \quad (4.11)$$

where  $\psi_A(q)$  and  $\psi_B(q)$  are introduced in eqs.(3.1)(3.3) respectively. This  $\rho$  satisfies  $\int_C dq\rho = 1$ , where  $C$  is an arbitrary contour running from  $-\infty$  to  $\infty$  in the complex  $q$ -plane. Then defining a probability current density  $j$  by

$$j(q, t) = \frac{\frac{i\hbar}{2m} \left( \frac{\partial\tilde{\psi}_B^{*q}}{\partial q}\psi_A - \tilde{\psi}_B^{*q}\frac{\partial\psi_A}{\partial q} \right)}{\langle B|A\rangle}, \quad (4.12)$$

we have the continuity equation  $\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial q}j(q, t) = 0$ . Therefore, probability interpretation seems to work formally with this  $\rho$ , although  $\rho$  is complex.

### §5. Correspondence principle to ordinary quantum mechanics

The future-included theory may look exotic because it includes time integration over not only the past but also the future. For such a theory to be viable it is very important to recapture the usual quantum mechanics even approximately from the future-included theory. Indeed, in ref.,<sup>3)</sup> such a possibility is speculated upon. We first examine the argument in ref.,<sup>3)</sup> and find that there are points to be improved. Next we study eq.(1.1) carefully by utilizing the proper inner product and the mechanism of suppressing the anti-Hermitian part of the Hamiltonian, and propose the correspondence principle of the future-included theory to ordinary quantum mechanics.

### 5.1. Former attempt to see the correspondence

We briefly review the speculation to obtain the correspondence in ref.<sup>3)</sup> and see that there are points to be improved. In ref.,<sup>3)</sup> it was suggested that eq.(1.1) is rewritten as

$$\langle \mathcal{O} \rangle^{BA} = \frac{\langle A(t)|B(t) \rangle \langle B(t)|\mathcal{O}|A(t) \rangle}{\langle A(t)|B(t) \rangle \langle B(t)|A(t) \rangle}, \quad (5.1)$$

and an attempt was made to approximate  $|B(t)\rangle\langle B(t)|$  as

$$|B(t)\rangle\langle B(t)| \simeq \frac{1}{N} \sum_w |w\rangle\langle w| = \frac{1}{N} 1 \quad (5.2)$$

for any  $t$  except for times only slightly later than the early Big Bang time, where  $N$  denotes the number of some orthonormal basis states  $|w\rangle$  ( $w = 1, 2, \dots, N$ ) such that  $\langle w|w'\rangle = \delta_{w,w'}$ , with the assumption that the system is sufficiently ergodic. If we admit that eq.(5.2) is a good approximation, then  $\langle \mathcal{O} \rangle^{BA}$  becomes the expectation value in the future-not-included theory,

$$\langle \mathcal{O} \rangle^{BA} \simeq \langle \mathcal{O} \rangle^{AA}, \quad (5.3)$$

where  $\langle \mathcal{O} \rangle^{AA}$  is given in eq.(1.2). But, is eq.(5.2) really a good approximation?

Eq.(5.2) cannot be true at all  $t$ . In fact, using eq.(3.5) we obtain

$$\begin{aligned} \frac{d}{dt} (|B(t)\rangle\langle B(t)|) &= -\frac{i}{\hbar} [\hat{H}_B^h, |B(t)\rangle\langle B(t)|] - \frac{i}{\hbar} \{ \hat{H}_B^a, |B(t)\rangle\langle B(t)| \} \\ &\simeq -2\frac{i}{\hbar} \hat{H}_B^a, \end{aligned} \quad (5.4)$$

where  $\hat{H}_B^h = \frac{\hat{H}_B + \hat{H}_B^\dagger}{2}$  and  $\hat{H}_B^a = \frac{\hat{H}_B - \hat{H}_B^\dagger}{2}$ , and in the second equality we have used eq.(5.2). Thus we have  $|B(t)\rangle\langle B(t)| \simeq C_1 \exp \left[ -2\frac{i}{\hbar} \hat{H}_B^a t \right]$ , where  $C_1$  is some constant. If we choose  $C_1$  so that we have  $|B(t)\rangle\langle B(t)| \simeq \frac{1}{N} \exp \left[ -2\frac{i}{\hbar} (t - T_B) \hat{H}_B^a \right]$ , then for  $t = T_B$  this becomes  $|B(T_B)\rangle\langle B(T_B)| \simeq \frac{1}{N} 1$ . So eq.(5.2) becomes a good approximation near the far future time  $T_B$ , but it is not so good for general time  $t$ .

### 5.2. Our analysis of $\langle \mathcal{O} \rangle^{BA}$

We analyze eq.(5.1) more carefully by utilizing the proper inner product and the automatic hermiticity mechanism. Expanding  $|B(T_B)\rangle$  as  $|B(T_B)\rangle = \sum_i b_i |\lambda_i\rangle_B$ , we obtain

$$\begin{aligned} |B(t)\rangle\langle B(t)| &= e^{-i\hat{H}_B(t-T_B)} |B(T_B)\rangle\langle B(T_B)|_{Q_B} e^{i\hat{H}_B^{QB}(t-T_B)} Q_B^{-1} \\ &= \sum_{i,j} b_i b_j^* e^{i\text{Re}(\lambda_j - \lambda_i)(t-T_B)} e^{i\text{Im}(\lambda_j + \lambda_i)(T_B - t)} |\lambda_i\rangle_B \langle \lambda_j| \\ &\simeq \frac{\int_{t-\Delta t}^{t+\Delta t} |B(t)\rangle\langle B(t)| dt}{\int_{t-\Delta t}^{t+\Delta t} dt} \end{aligned}$$

$$\begin{aligned}
&\simeq \sum_i |b_i|^2 e^{2\text{Im}(\lambda_i)(T_B-t)} |\lambda_i\rangle_B \langle \lambda_i| \\
&\simeq e^{2B(T_B-t)} Q_2 \quad \text{for large } T_B - t,
\end{aligned} \tag{5.5}$$

where  $Q_B$  and  $\hat{H}_B^{\dagger Q_B}$  are given in eqs.(3.12)(3.13) respectively. In the third line we have smeared the present time  $t$  a little bit, and then, since the off-diagonal elements wash to 0, we are led to the fourth line. In the last line we have used the automatic hermiticity mechanism for large  $T_B - t$ , and  $Q_2$  is given by

$$\begin{aligned}
Q_2 &= \sum_{i \in A} |b_i|^2 |\lambda_i\rangle_B \langle \lambda_i| \\
&= \sum_{i \in A} F(\hat{H}_{\text{eff}}^B) |\lambda_i\rangle_B \langle \lambda_i| \\
&= F(\hat{H}_{\text{eff}}^\dagger) Q \quad \text{for the restricted subspace,}
\end{aligned} \tag{5.6}$$

where in the second equality, assuming that the  $\text{Re}\lambda_i$  are not degenerate, we have interpreted  $|b_i|^2$  as a function of  $\text{Re}\lambda_i$ ,  $|b_i|^2 = F(\text{Re}\lambda_i)$ . Also,  $\hat{H}_{\text{eff}}^B \equiv P_B \tilde{D}_R P_B^{-1} = \hat{H}_{\text{eff}}^\dagger$  is  $Q_B$ -Hermitian,  $(\hat{H}_{\text{eff}}^B)^{\dagger Q_B} = \hat{H}_{\text{eff}}^B$ , and obeys  $\hat{H}_{\text{eff}}^B |\lambda_i\rangle_B = \text{Re}\lambda_i |\lambda_i\rangle_B$ . In the last equality we have utilized the relation  $\sum_{i \in A} |\lambda_i\rangle \langle \lambda_i|_Q = 1$  for the subspace restricted by the subgroup  $A$ .

Next we expand  $|A(t)\rangle$  as  $|A(t)\rangle \equiv \sum_i a_i(t) |\lambda_i\rangle$ , and define  $|\tilde{A}(t)\rangle \equiv \sum_{i \in A} a_i(t) |\lambda_i\rangle$ . Then for large  $t - T_A$ , since we have  $|A(t)\rangle \simeq |\tilde{A}(t)\rangle$  by using the automatic hermiticity mechanism as in eq.(3.17), we can express eq.(5.1) as

$$\langle \mathcal{O} \rangle^{BA} \simeq \frac{\langle \tilde{A}(t) |_{Q_2} \mathcal{O} | \tilde{A}(t) \rangle}{\langle \tilde{A}(t) |_{Q_2} \tilde{A}(t) \rangle} \quad \text{for large } T_B - t \text{ and large } t - T_A, \tag{5.7}$$

where  $Q_2$  is given by eq.(5.6). In eq.(5.7)  $|\tilde{A}(t)\rangle$  is really the state of our whole universe as obtained from the initial state. From a classical point of view it is likely to be a superposition of many wildly different states representing narrow wave packets. In practice, since we live inside this universe, we come to know features that in the CAT are determined from  $\langle B(t) |$ , i.e. “the future”, rather than only from  $|\tilde{A}(t)\rangle$ . Information about such features coming from the future may partly stand in our memories and we can combine this information with information on  $|\tilde{A}(t)\rangle$  to obtain a better and in some way more realistic replacement for  $|\tilde{A}(t)\rangle$  which we call  $|\tilde{\psi}_{\text{memory}}(t)\rangle$ . We hope to return to this improvement of  $|\tilde{A}(t)\rangle$  to  $|\tilde{\psi}_{\text{memory}}(t)\rangle$  in a later article, but since this replacement plays effectively no important role in the present article, we shall just keep the notation  $|\tilde{A}(t)\rangle$  and bear in mind that it would be more realistic to call it  $|\tilde{\psi}_{\text{memory}}(t)\rangle$ .

### 5.3. Our proposal of the correspondence

In ref.<sup>3)</sup> an expectation value in a future-not-included theory is defined by eq.(1.2), but in refs.<sup>13), 14)</sup> and in eq.(3.21) we defined a slightly different one by

$$\langle \mathcal{O} \rangle_Q^{AA} \equiv \frac{\langle A(t)|_Q \mathcal{O} |A(t)\rangle}{\langle A(t)|_Q A(t)\rangle}. \quad (5.8)$$

In this subsection we show that it is the latter definition of an expectation value that we obtain from a future-included theory. We first point out that the operator  $P' = Pf(D)$ , where  $f(D)$  is some function of  $D$ , is another diagonalizing matrix of  $\hat{H}$ , because  $P'DP'^{-1} = PDP^{-1} = \hat{H}$ . So we can define another inner product with  $Q' = (P'^{\dagger})^{-1}P'^{-1}$ . Choosing the function  $f$  such that  $(P^{\dagger})^{-1}f(DD^{\dagger})^{-1}P^{\dagger} = F(\hat{H}^{\dagger})$ , we obtain  $Q' = F(\hat{H}^{\dagger})Q \simeq F(\hat{H}_{\text{eff}}^{\dagger})Q = Q_2$  for the restricted subspace, where we have utilized the automatic hermiticity mechanism for large  $t - T_A$ , for which  $|A(t)\rangle$  is replaced with  $|\tilde{A}(t)\rangle$ . Then with the inner product  $I_{Q'}$  the expectation value in a future-not-included theory is expressed as

$$\begin{aligned} \langle \mathcal{O} \rangle_{Q'}^{AA} &= \frac{\langle A(t)|_{Q'} \mathcal{O} |A(t)\rangle}{\langle A(t)|_{Q'} A(t)\rangle} \\ &\simeq \frac{\langle \tilde{A}(t)|_{Q_2} \mathcal{O} |\tilde{A}(t)\rangle}{\langle \tilde{A}(t)|_{Q_2} \tilde{A}(t)\rangle} \quad \text{for large } t - T_A. \end{aligned} \quad (5.9)$$

Comparing eq.(5.7) with eq.(5.9), we obtain the following correspondence:

$$\langle \mathcal{O} \rangle^{BA} \text{ for large } T_B - t \text{ and large } t - T_A \quad \simeq \quad \langle \mathcal{O} \rangle_{Q'}^{AA} \text{ for large } t - T_A. \quad (5.10)$$

This relation means that the future-included theory for large  $T_B - t$  and large  $t - T_A$  is almost equivalent to the future-not-included theory with the proper inner product for large  $t - T_A$ . This equivalence suggests that the future-included theory is *not excluded*, although it seems exotic.

### 5.4. A seemingly time reversal symmetry problem

Finally, let us discuss a seemingly time reversal symmetry problem. In the functional integral formulation with future included there seems to be no difference between the past and future time directions. In the light of this, it seems strange that we obtain a description in terms of the form  $\langle \mathcal{O} \rangle_{Q'}^{AA}$  rather than the form  $\langle \mathcal{O} \rangle_{Q''}^{BB}$  with another appropriate operator  $Q''$  which means that we use the future instead of the past. Essentially  $\langle \mathcal{O} \rangle^{BA}$  can be rewritten as either  $\langle \mathcal{O} \rangle_{Q'}^{AA}$  or  $\langle \mathcal{O} \rangle_{Q''}^{BB}$ , as we like. However, we know phenomenologically that the past state influences the present state, while the future state does not influence the present state so much, so we can choose the expression  $\langle \mathcal{O} \rangle_{Q'}^{AA}$ . But we hope that we can explain why it is chosen. Our present universe is at a low temperature and has a low energy density, while

the imaginary part of the action is very small. On the other hand, the situation of an early universe – high temperature and high energy density – is very different from our present era. So we have some possibility that in the early universe there was a period in which the imaginary part of the Lagrangian was much more important than later or in the future. This possibility is speculated on in refs.<sup>3)–5)</sup> It is an open problem to show this explicitly, but if it is proven to be true, then the reason that the present expectation value is to be described approximately in terms of a model with an initial state determined picture meaning  $\langle \mathcal{O} \rangle_{Q'}^{AA}$  would be that the solution is mainly determined by a requirement involving the imaginary part of the Lagrangian in the early universe, while at most small clear signals come from the future. Thus  $\langle \mathcal{O} \rangle^{BA}$  should be rewritten as  $\langle \mathcal{O} \rangle_{Q'}^{AA}$  rather than  $\langle \mathcal{O} \rangle_{Q''}^{BB}$ . We speculate that the past and the future are physically different in the sense that the past is  $S_I$ -dominant, while the future is not  $S_I$ -dominant, which essentially causes the time reversal symmetry breaking.

## §6. Summary and outlook

We have studied a future-included version of a complex action theory (CAT), which includes time integration over not only past but also future. In ref.<sup>3)</sup> a correspondence of the theory to ordinary quantum mechanics, i.e. a future-not-included version of a real action theory (RAT), was speculated upon. In this paper, studying the quantity  $\langle \mathcal{O} \rangle^{BA}$  defined in eq.(1.1) more carefully and in detail using both the automatic hermiticity mechanism<sup>13),14)</sup> and the complex coordinate formalism,<sup>14)</sup> we have confirmed that, even if future is fundamentally included in the formalism, it leads to only minute deviations from ordinary quantum mechanics. This correspondence principle is one of the main results obtained in this paper.

In section 2 we reviewed our complex coordinate formalism and gave a theorem for matrix elements of the form  ${}_m\langle_{new} q' \text{ or } p' | \mathcal{O}(\hat{q}_{new}, \hat{q}_{new}^\dagger, \hat{p}_{new}, \hat{p}_{new}^\dagger) | q'' \text{ or } p'' \rangle_{new}$ , which states that we can ignore the anti-Hermitian terms in  $\hat{q}_{new}$ ,  $\hat{q}_{new}^\dagger$ ,  $\hat{p}_{new}$  and  $\hat{p}_{new}^\dagger$  provided that we are satisfied with the result in the distribution sense. In section 3, after explaining the two states  $\langle B(t) |$  and  $| A(t) \rangle$  definable from their respective functional integrals over future and past following ref.,<sup>3)</sup> we gave the two slightly improved wave functions  $\psi_A(q) = {}_m\langle_{new} q | A(t) \rangle$  and  $\psi_B(q) = {}_m\langle_{new} q | B(t) \rangle$  based on the complex coordinate formalism,<sup>14)</sup> so that they are properly defined even for complex  $q$ . Then, reviewing the proper inner product for the Hamiltonian  $\hat{H}$  in the future-not-included theory, we constructed the proper inner product for the other Hamiltonian  $\hat{H}_B$  in the future-included theory. We also reviewed the automatic hermiticity mechanism for  $\hat{H}$  in the future-not-included theory. In section 4 we studied the behavior of  $\langle \mathcal{O} \rangle^{BA}$ . We showed that it allows us to have the Heisenberg equation and

Ehrenfest's theorem. We also obtained the momentum relation  $\langle \hat{p}_{new} \rangle^{BA} = m \frac{d}{dt} \langle \hat{q}_{new} \rangle^{BA}$  in eq.(4.8), which is consistent with the result of ref.<sup>15)</sup> Furthermore, we constructed a conserved probability current density by using both  $|A\rangle$  and  $|B\rangle$ . We have thus checked that  $\langle \mathcal{O} \rangle^{BA}$  provides the saddle point development with  $t$  and works as an expectation value though its appearance is a matrix element.

In section 5 we analyzed the quantity  $\langle \mathcal{O} \rangle^{BA}$ , and derived the correspondence of the future-included theory to ordinary quantum mechanics. Showing that usual physics is approximately obtained from the future-included theory is very important, because the future-included theory seems excluded phenomenologically from two unusual points: the existence of the imaginary part of the action  $S$  and that of the future state. We first reviewed the speculation on the correspondence in ref.,<sup>3)</sup> and made it clear that there are points to be improved in the argument. Then we studied  $\langle \mathcal{O} \rangle^{BA}$  with more care concerning the inner product being obtained by using both the complex coordinate formalism and the automatic hermiticity mechanism, and showed that the quantity  $\langle \mathcal{O} \rangle^{BA}$  becomes an expectation value with a different inner product defined in a future-not-included theory. Next we showed that the inner product can be interpreted as one of the possible proper inner products realized in the future-not-included theory. Thus we have obtained the correspondence principle in eq.(5.10), i.e.  $\langle \mathcal{O} \rangle^{BA}$  for large  $t - T_A$  and large  $T_B - t \simeq \langle \mathcal{O} \rangle_{Q'}^{AA}$  for large  $t - T_A$ , where  $T_A$ ,  $T_B$ , and  $t$  are the past initial time, the future final time, and the present time, respectively, and  $Q'$  is a Hermitian operator used to define the proper inner product. This relation means that the future-included theory for large  $T_B - t$  and large  $t - T_A$  is almost equivalent to the future-not-included theory with the proper inner product for large  $t - T_A$ .

Thus it is not excluded that fundamentally the action is complex. Indeed, the reality of an action in ordinary quantum theory can be regarded as a restriction on parameters in the action, so it is a benefit of our theory that we can have a more general action by getting rid of the restriction. Also, since we found that the effects of backward causation are in practice small, a theory with a functional integral of future time is not excluded and what happens in the future can actually in principle act back on us today. In addition, as we have seen in this paper, the future-included theory looks more elegant in the functional integral formulation and shows more cleanly the saddle points providing the classical approximation than the future-not-included theory that we studied previously in refs.,<sup>13), 14)</sup> where we encountered additional complicated terms. Furthermore, the future-included theory can yield the future-not-included theory with the proper inner product. These are the advantages of the future-included theory.

As our next steps, what should we study? First we note that in the above correspondence we have the Hermitian operator  $Q'$ . It is a priori non-local, but phenomenologically it

should be local. We expect that it becomes effectively local somehow in some reasonable approximation. It would be desirable to invent some mechanism for making it effectively local. Also, it is very important to study the dynamics in some concrete model of the future-included theory. Indeed, since we already have the complex coordinate formalism and so on, it would be possible to perform the analyses. Furthermore, a transactional interpretation<sup>26)</sup> of quantum mechanics is discussed in refs.<sup>7)8)</sup> based on the future-included theory. It would be interesting to study in detail the relation between the interpretation and the future-included theory. We hope to report studies on these problems in the future.

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### Appendix A

#### —— Derivation of $\hat{H}_B$ ——

The Feynman path integral (FPI) in the complex action theory (CAT) is described with the following Lagrangian, with a single degree of freedom for simplicity:

$$L(q(t), \dot{q}(t)) = \frac{1}{2}m\dot{q}^2 - V(q), \quad (\text{A}\cdot 1)$$

where  $V(q)$  is a potential term defined by  $V(q) = \sum_{n=2}^{\infty} b_n q^n$ . This Lagrangian has the same form as that in the real action theory (RAT), but since we consider it in the CAT,  $m$ ,  $q$  and any other parameters included in  $V(q)$  are complex in general. We consider the integrand of the FPI  $\exp\left(\frac{i}{\hbar} \int L dt\right)$  by discretizing the time direction and writing  $\dot{q}$  as  $\dot{q} = \frac{q(t+dt) - q(t)}{dt}$ , where  $dt$  is assumed to be a small quantity. Since we use the Schrödinger representation for wave functions, to avoid confusion with the Heisenberg representation we introduce the notations  $q_t \equiv q(t)$  and  $q_{t+dt} \equiv q(t + dt)$ , which we regard as independent variables.

In ref.<sup>15)</sup> we explicitly examined the momentum and the Hamiltonian since it is not trivial whether we can use the same forms as those in the RAT, because they includes a quantity at time  $t + dt$ ,  $q_{t+dt}$ , which is somehow unclear from the point of view of quantum mechanics unless we define it properly including the fluctuation in the time development from a quantity at time  $t$ . Thus we are motivated to examine them by describing  $q_{t+dt}$  properly via FPI. We briefly explain how we derived  $\hat{H}$  in ref.<sup>15)</sup>



In FPI, the time development of some wave function  ${}_m\langle_{new} q_t|\psi(t)\rangle$  at time  $t$  to  $t + dt$  is described by

$${}_m\langle_{new} q_{t+dt}|\psi(t + dt)\rangle = \int_C e^{\frac{i}{\hbar}\Delta t L(q,\dot{q})} {}_m\langle_{new} q_t|\psi(t)\rangle dq_t, \quad (\text{A}\cdot 2)$$

where  $L(q, \dot{q})$  is given by eq.(A.1), and  $C$  is an arbitrary path running from  $-\infty$  to  $\infty$  in the complex plane. In ref.,<sup>15)</sup> to derive the momentum relation  $p = \frac{\partial L}{\partial \dot{q}}$ , we considered some wave function  ${}_m\langle_{new} q_t|\xi\rangle$  that obeys

$$\begin{aligned} {}_m\langle_{new} q_t|\hat{p}_{new}|\xi\rangle &= \frac{\hbar}{i} \frac{\partial}{\partial q_t} {}_m\langle_{new} q_t|\xi\rangle \\ &= \frac{\partial L}{\partial \dot{q}} \left( q_t, \frac{\xi - q_t}{dt} \right) {}_m\langle_{new} q_t|\xi\rangle, \end{aligned} \quad (\text{A}\cdot 3)$$

where  $\xi$  is any number. Since the set  $\{|\xi\rangle\}$  is an approximately reasonable basis that has roughly completeness  $1 \simeq \int_C d\xi |\xi\rangle {}_m\langle_{anti} \xi|$  and orthogonality  ${}_m\langle_{anti} \xi|\xi'\rangle \simeq \delta_c(\xi - \xi')$ , where  ${}_m\langle_{anti} \xi|$  is a dual basis of  $|\xi\rangle$ , we can expand the wave function  ${}_m\langle_{new} q_t|\psi(t)\rangle$  into a linear combination of  ${}_m\langle_{new} q_t|\xi\rangle$  as  ${}_m\langle_{new} q_t|\psi(t)\rangle = \int_C d\xi {}_m\langle_{new} q_t|\xi\rangle {}_m\langle_{anti} \xi|\psi(t)\rangle$ . Then, solving eq.(A.3), we can estimate the right-hand side of eq.(A.2) explicitly as follows:

$$\begin{aligned} {}_m\langle_{new} q_{t+dt}|\psi(t + dt)\rangle &= \int'_C d\xi \int_C dq_t e^{\frac{i}{\hbar}\Delta t L(q,\dot{q})} {}_m\langle_{new} q_t|\xi\rangle {}_m\langle_{anti} \xi|\psi(t)\rangle \\ &\simeq \sqrt{\frac{2\pi i \hbar dt}{m}} {}_m\langle_{new} q_{t+dt}|\exp\left(-\frac{i}{\hbar}\hat{H}dt\right)|\psi(t)\rangle, \end{aligned} \quad (\text{A}\cdot 4)$$

where  $\hat{H}$  is given by eq.(3.7). Thus we have found that the Hamiltonian  $\hat{H}$  has the same form as that in the RAT starting from eq.(A.2).

We can obtain the expression of  $\hat{H}_B$  analogously to the calculation in ref.<sup>15)</sup> just by noticing the following points. Performing the  $*_q$  operation on eq.(A.2) we obtain  $\langle\psi(t + dt)|q_{t+dt}\rangle_{new} = \int_C e^{-\frac{i}{\hbar}\Delta t L(q,\dot{q})*_q} \langle\psi(t)|q_t\rangle_{new} dq_t$ . Defining  $dt' = -dt$ , we rewrite this as

$$\langle\psi(t - dt')|q_{t-dt'}\rangle_{new} = \int_C e^{\frac{i}{\hbar}\Delta t' L(q,\dot{q})*_q} \langle\psi(t)|q_t\rangle_{new} dq_t, \quad (\text{A}\cdot 5)$$

where  $L(q, \dot{q})*_q = \frac{1}{2}m^*\dot{q}^2 - V(q)*_q$  and  $V(q)*_q = \sum_{n=2} b_n^* q^n$ . On the other hand, the time development of the wave function  $\langle B(t)|q_t\rangle$  at time  $t$  to time  $t - dt$  is described by

$$\langle B(t - \Delta t)|q'_{t-\Delta t}\rangle_{new} = \int_{\text{path}(t-\Delta t)=q'_{t-\Delta t}} \langle B(t)|q_t\rangle_{new} e^{\frac{i}{\hbar}S_{t-\Delta t} \text{ to } t} D\text{path}. \quad (\text{A}\cdot 6)$$

Comparing this expression with eq.(A.5), we can derive  $\hat{H}_B$  in a similar way to the derivation of  $\hat{H}$  in ref.<sup>15)</sup> Indeed,  $\hat{H}_B$  is given just by the replacement of the coupling parameters and operators with their complex or Hermitian conjugates in the expression of  $\hat{H}$ ,  $\hat{H}_B = \frac{1}{2m^*}(\hat{p}_{new}^\dagger)^2 + \sum_{n=2} b_n^*(\hat{q}_{new}^\dagger)^n = \hat{H}^\dagger$ . Thus we have derived eq.(3.6).

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